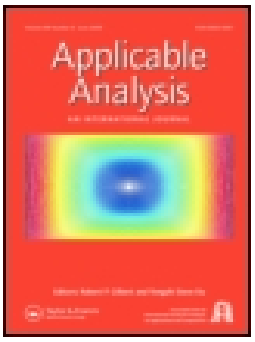

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Tikhonov regularization method for a backward problem for the inhomogeneous time-fractional diffusion equation

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ABSTRACT

Fractional (nonlocal) diffusion equations replace the integer-order derivatives in space and time by their fractional-order analogs and they are used to model anomalous diffusion, especially in physics. In this paper, we study a backward problem for an inhomogeneous time-fractional diffusion equation with variable coefficients in a general bounded domain. Such a backward problem is of practically great importance because we often do not know the initial density of substance, but we can observe the density at a positive moment. The backward problem is ill-posed and we propose a regularizing scheme by using Tikhonov regularization method. We also prove the convergence rate for the regularized solution by using an a priori regularization parameter choice rule. Numerical examples illustrate applicability and high accuracy of the proposed method.

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1. Introduction

Partial differential equations of fractional orders have recently become a focus of many research studies because of their various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. Fractional calculus in mathematics is a natural extension of integer-order calculus. One well-known model in slow diffusion for the inhomogeneous time-fractional diffusion equation is given by the following problem:

$$\begin{cases} D_t^{\gamma} u = \mathcal{L}u(x,t) + G(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^d$ is a bounded domain with sufficient smooth boundary $\partial\Omega$, $G(x,t)$ is a given source function in $\Omega \times (0,T)$, $\gamma \in (0,1)$ is the fractional order of the time derivative, D_t^{γ} is the (left-sided) Caputo fractional time derivative defined by

$$D_t^{\gamma} u := \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} u'(s) \mathrm{d}s,$$

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and $T > 0$ is a final time. Note that if $\gamma = 1$ and $\gamma = 2$ then the equation in (1.1) represents a parabolic equation and a hyperbolic equation, respectively. Let $L^2(\Omega)$ be a usual L^2- space with the scalar product $< \cdot, \cdot >$ and $H^l(\Omega)$, $H_0^m(\Omega)$ denote the Sobolev spaces, see [1]. Let $\mathcal{L}$ be given by

$$\mathcal{L}v(x) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} A_{ij}(x)\frac{\partial}{\partial x_j}v(x)\right) + C(x)v(x), \quad x \in \Omega,$$

where $A_{ij} = A_{ji}$. Moreover, we assume that the operator $-\mathcal{L}$ is uniformly elliptic operator defined on $D(-\mathcal{L}) = H^2(\Omega) \cap H_0^1(\Omega)$ and that its coefficients are smooth: there exists a constant ν such that

$$\nu \sum_{i=1}^{d} \xi_i^2 \le \sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij}(x)\,\xi_i\,\xi_j, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d,$$

and the coefficient functions satisfy

$$A_{ij} \in C^1(\overline{\Omega}), \quad C \in C(\overline{\Omega}), \quad C(x) \le 0, \quad x \in \overline{\Omega}.$$

For given inputs γ, $G(x,t)$ and $u_0(x)$, the problem (1.1) is called the direct (forward) problem. Like most direct problems of the mathematical physics, the problem (1.1) is well posed. In [2], it is proved that the direct problem (1.1) has a unique weak solution.

As it is known, a direct problem aims to find a solution that satisfies given differential equation (ordinary, partial, or fractional) and related to initial and boundary conditions. In some problems, the main equation and the conditions are not sufficient to obtain the solution, but, instead some additional conditions (also called measured output data) are required. Such problems are called the inverse problems. Recently, there has been a growing interest in inverse problems with fractional derivatives. Usually, in these works a fractional time derivative is considered and determination of that, a coefficient function or a source term under some additional condition(s) is the inverse problem. These problems are physically and practically very important. We list some of the important references [2–14]. This study can be regarded as continuation of the series of works mentioned above on fractional inverse problems.

In this paper, we study a backward determination problem. The backward problem of diffusion process is of great importance in engineering and aims at detecting the previous status of physical field from its present information. In general, no solution which satisfies the diffusion equation with final data and the boundary conditions exists. Even if a solution exists, it does not depend continuously on the final data and any small perturbation in the given data may cause large change to the solution which means the backward problem is ill-posed in the sense of Hadamard. So we need some regularization methods to deal with this problem. We construct a regularized solution of the backward problem by using Tikhonov regularization method and present stability analysis and error estimate.

Although the backward problem for the homogeneous time-fractional diffusion equation i.e. $D_t^\gamma u = \mathcal{L}u(x,t)$ has been studied in the mathematical literature [14–16], the backward problem for the inhomogeneous time-fractional diffusion equation i.e. $D_t^\gamma u = \mathcal{L}u(x,t) + G(x,t)$ has not yet been studied. Motivated by this reason, we consider the following backward determination problem:

$$\begin{cases} D_t^\gamma u = \mathcal{L}u(x,t) + G(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,T) = g(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The backward problem here consists of determining $u(x,t)$ for $t \in [0,T)$ in the problem (1.2) from given inputs $G(x,t)$ and $g(x)$. However, in practice the measurements can be only given with some noise level (measurement error) ϵ. Hence, for the inputs $G(x,t)$ and $g(x)$ of the backward problem, we assume that

$$\|g^\epsilon - g\|_{L^2(\Omega)} \le \epsilon, \quad \|G^\epsilon - G\|_{L^\infty(0,T;L^2(\Omega))} \le \epsilon. \tag{1.3}$$

Therefore the backward problem is to find $u(x,t)$ for $t \in [0,T)$ in the problem (1.2) from given inputs $\left(g^\epsilon, G^\epsilon\right)$ such that (1.3) holds. Here and afterwards the $L^\infty(0,T;X)$ consists of all measurable functions $u : [0,T] \to X$ with $\|u\|_{L^\infty(0,T;X)} := \text{ess sup}_{0\le t\le T}\|u(t)\|_X$, where X denotes a real Banach space with the norm $\|.\|_X$.

The paper is organized as follows. In Section 2, we provide some preliminary material. The existence, uniqueness, and regularity of the solution for the backward problem are obtained in Section 3. In Section 4, we propose a Tikhonov regularization method and give a convergence estimate under an a priori assumption. Two numerical examples are given to show the effectiveness of our method in Section 5.

2. Preliminaries

This section is devoted to some useful definitions and lemmas.

Definition 2.1 [17]: The Mittag–Leffler function is defined as

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \quad z \in \mathbb{C}, \tag{2.1}$$

where $\gamma > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

We note that, by the power series, we can directly verify that $E_{\gamma,\beta}(z)$ is an entire function of $z \in \mathbb{C}$.

The following lemmas indicate three important properties of the Mittag–Leffler function which provide technical convenience in ensuing theorems ahead.

Lemma 2.1 [14]: *Let $0 < \gamma_0 < \gamma_1 < 1$. Then there exists constants $B^-, B^+ > 0$ depending only on γ_0, γ_1 such that*

$$\frac{B^-}{\Gamma(1-\gamma)}\frac{1}{1-x} \le E_{\gamma,1}(x) \le \frac{B^+}{\Gamma(1-\gamma)}\frac{1}{1-x}, \quad \forall x \le 0.$$

These estimates are uniform for all $\gamma \in [\gamma_0, \gamma_1]$.

Unless otherwise specified, we assume $\gamma \in [\gamma_0, \gamma_1]$ for some γ_0, γ_1 in this paper.

Lemma 2.2 [17]: *Let $\lambda > 0$. Then we have*

$$D_t^\gamma E_{\gamma,1}(-\lambda t^\gamma) = -\lambda E_{\gamma,1}(-\lambda t^\gamma), \quad t > 0, \; 0 < \gamma < 1. \tag{2.2}$$

Lemma 2.3 [17]: *The following equality holds for $\lambda > 0$, $\gamma > 0$, and $n \in \mathbb{N}$:*

$$\frac{d^n}{dt^n}E_{\gamma,1}(-\lambda t^\gamma) = -\lambda t^{\gamma-n}E_{\gamma,\gamma-n+1}(-\lambda t^\gamma), \quad t > 0. \tag{2.3}$$

Lemma 2.4: *Let $m, \alpha > 0, A, z_0$ be positive numbers. If $z \in (0, +\infty)$ then there holds*

$$K(z) = \frac{\alpha z^{2-m}}{\alpha z^2 + A} \leq \begin{cases} \frac{1}{2} A^{-\frac{m}{2}} \left(2-m\right)^{\frac{2-m}{2}} (\alpha m)^{\frac{m}{2}}, & \text{if } 0 < m < 2, \quad z \in (0, +\infty), \\ A^{-1} z_0^{2-m} \alpha, & \text{if } \quad m \geq 2, \quad z \in (z_0, +\infty), \end{cases} \tag{2.4}$$

and

$$F(z) = \frac{\alpha z^{1-m}}{\alpha z^2 + A} \leq \begin{cases} \frac{1}{2} \left(\frac{1}{A}\right)^{\frac{m+1}{2}} (1-m)^{\frac{1-m}{2}} (1+m)^{\frac{1+m}{2}} \alpha^{\frac{1+m}{2}}, & \text{if } 0 < m < 1, \quad z \in (0, +\infty), \\ A^{-1} z_0^{1-m} \alpha, & \text{if } \quad m \geq 1, \quad z \in (z_0, +\infty). \end{cases} \tag{2.5}$$

***Proof*:** First we prove (2.4).

Case 1: $0 < m < 2$. It is easy to see that $\lim_{z \to 0} K(z) = \lim_{z \to \infty} K(z) = 0$. Thus

$$K(z) \leq \sup_{z \in (0, +\infty)} K(z) \leq K(z^*), \quad \text{for any} \quad z \in (0, +\infty),$$

where $z^* \in (0, +\infty)$ such that $K'(z^*) = 0$. By a simple calculation, we find $z^* = \left(\frac{A(2-m)}{m\alpha}\right)^{\frac{1}{2}} > 0$. This leads us to

$$K(z) \leq K(z^*) = \frac{\alpha \left(\frac{A(2-m)}{m\alpha}\right)^{\frac{2-m}{2}}}{A + \frac{A(2-m)}{m}} = \frac{1}{2} A^{-\frac{m}{2}} \left(2-m\right)^{\frac{2-m}{2}} (\alpha m)^{\frac{m}{2}}.$$

Case 2: $m \geq 2$. Since $z \geq z_0$ and $m \geq 2$, we have $z^{2-m} \leq z_0^{2-m}$. This implies that

$$K(z) = \frac{\alpha z^{2-m}}{\alpha z^2 + A} \leq \frac{\alpha z^{2-m}}{A} \leq A^{-1} z_0^{2-m}.$$

Next we prove (2.5).

Case 1: $0 < m < 1$. It is obvious that $\lim_{z \to 0} F(z) = \lim_{z \to \infty} F(z) = 0$. Thus

$$F(z) \leq \sup_{z \in (0, +\infty)} F(z) \leq F(z^+),$$

where $z^+ \in (0, +\infty)$ such that $F'(z^+) = 0$. By a simple calculation, we find $z^+ = \left(\frac{A(1-m)}{\alpha(1+m)}\right)^{\frac{1}{2}} > 0$. This leads us to

$$F(z) \leq F(z^+) = \frac{\left(\frac{A(1-m)}{(1+m)\alpha}\right)^{\frac{1-m}{2}} \alpha}{\frac{A(1-m)}{1+m} + A} = \frac{1}{2} \left(\frac{1}{A}\right)^{\frac{m+1}{2}} (1-m)^{\frac{1-m}{2}} (1+m)^{\frac{1+m}{2}} \alpha^{\frac{1+m}{2}}.$$

Case 2: $m \geq 1$. It is clear to see that $z^{1-m} \leq z_0^{1-m}$ from $z \geq z_0$ and $m \geq 1$. Hence, we get

$$F(z) = \frac{\alpha z^{1-m}}{\alpha z^2 + A} \leq \frac{\alpha z^{1-m}}{A} \leq A^{-1} z_0^{1-m} \alpha.$$

□

In the next sections, we will need the solution of the direct problem (1.1). For this purpose, we give a well-known solution formula for the direct problem (1.1). Since $-\mathcal{L}$ is a symmetric uniformly elliptic operator, the eigenvalues of $-\mathcal{L}$ satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_p \leq \cdots$$

with $\lambda_p \to \infty$ as $p \to \infty$. We denote the corresponding eigenfunctions by $\varphi_p \in H^2(\Omega) \cap H_0^1(\Omega)$. Thus the pair (λ_p, φ_p), $p = 1, 2, \ldots$, satisfies

$$\begin{cases} \mathcal{L}\phi_p(x) = -\lambda_p \phi_p(x), & x \in \Omega, \\ \varphi_p(x) = 0, & x \in \partial\Omega. \end{cases}$$

The functions ϕ_p are normalized so that $\{\phi_p\}_{p=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. Then we have

$$H^k(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{p=1}^{\infty} \lambda_p^{2k} \left| \langle v, \phi_p \rangle \right|^2 < +\infty \right\}.$$

We note that $H^k(\Omega)$ is a Hilbert space equipped with norm

$$\|v\|_{H^k(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_p^{2k} \left| \langle v, \phi_p \rangle \right|^2 \right)^{1/2}.$$

By using Fourier series expansion, the formal solution of the direct problem (1.1) can be found in the following form [2]:

$$u(x,t) = \sum_{p=1}^{\infty} \left(E_{\gamma,1}(-\lambda_p t^{\gamma}) u_{0,p} + \int_0^t (t-\tau)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-\tau)^{\gamma}) G_p(\tau) \mathrm{d}\tau \right) \phi_p(x), \quad (2.6)$$

where $u_{0,p} = \langle u_0(x), \phi_p(x) \rangle$ and $G_p(t) = \langle G(x,t), \phi_p(x) \rangle$.

3. The backward time-fractional diffusion problem

We begin this section with the following useful lemma.

Lemma 3.1: *Let $G \in L^{\infty}(0,T;L^2(\Omega))$. Then there exists a positive constant M_1 such that*

$$\sum_{p=1}^{\infty} \left| \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^{\gamma}) G_p(s) \mathrm{d}s \right|^2 \leq M_1 \|G\|_{L^{\infty}(0,T;L^2(\Omega))}^2. \quad (3.1)$$

Furthermore, if $G \in L^{\infty}(0,T;H^2(\Omega))$ then there holds

$$\sum_{p=1}^{\infty} \lambda_p^2 \left| \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^{\gamma}) G_p(s) \mathrm{d}s \right|^2 \leq M_1 \|G\|_{L^{\infty}(0,T;H^2(\Omega))}^2. \quad (3.2)$$

Proof: By applying Lemma 2.3 for $n = 1$ and using the fact that $E_{\alpha,1}(0) = 1$ we have

$$\int_0^t s^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p s^\gamma)\mathrm{d}s = -\frac{1}{\lambda_p}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}E_{\gamma,1}(-\lambda_p s^\gamma)\mathrm{d}s = \frac{1}{\lambda_p}(1 - E_{\gamma,1}(-\lambda_p t^\gamma)), \text{ for any } t > 0. \tag{3.3}$$

Also Lemma 3.3 in [2] gives $E_{\gamma,1}(-\lambda_n t^\gamma) \geq 0$ so that $(t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma) \geq 0$ for all $t \geq 0$. Hence

$$0 \leq \int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)\mathrm{d}s \leq \frac{1}{\lambda_p}, \quad t \geq 0. \tag{3.4}$$

Noting that

$$|G_p(t)|^2 \leq \sum_{p=1}^\infty \left| < G(x,t), \phi_p(x) > \right|^2 \leq \|G\|^2_{L^\infty(0,T;L^2(\Omega))} \tag{3.5}$$

for $t \in [0, T]$ and using (3.4) we have

$$\begin{aligned} \left|\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)G_p(s)\mathrm{d}s\right| &\leq \int_0^t \left|(t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)G_p(s)\right|\mathrm{d}s \\ &\leq \|G\|_{L^\infty(0,T;L^2(\Omega))}\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)\mathrm{d}s \\ &\leq \frac{\|G\|_{L^\infty(0,T;L^2(\Omega))}}{\lambda_p}. \end{aligned} \tag{3.6}$$

This implies that

$$I_1 = \sum_{p=1}^\infty \left|\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)G_p(s)\mathrm{d}s\right|^2 \leq \|G\|^2_{L^\infty(0,T;L^2(\Omega))}\sum_{p=1}^\infty \frac{1}{\lambda_p^2}.$$

It is known that $\lambda_p \geq Qp^{2/d}$ for $p \in \mathbb{N}$, where Q is a positive constant independent of p, [18]. Hence for $0 < d < 4$ by letting

$$M_1 := \frac{1}{Q^2}\sum_{p=1}^\infty \frac{1}{p^{4/d}},$$

we prove

$$\sum_{p=1}^\infty \left|\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)G_p(s)\mathrm{d}s\right|^2 \leq M_1\|G\|^2_{L^\infty(0,T;L^2(\Omega))}. \tag{3.7}$$

The proof of (3.2) is similar. □

Throughout this paper, we assume that $0 < d < 4$.

Theorem 3.1:

(a) *Let $g \in L^2(\Omega)$ and $G \in L^\infty(0,T;L^2(\Omega))$. Then the problem (1.2) has a unique solution $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ which is given by*

$$u(x,t)=\sum_{p=1}^{\infty}\left[\frac{E_{\gamma,1}(-\lambda_p t^\gamma)}{E_{\gamma,1}(-\lambda_p T^\gamma)}\left(g_p-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right)\right]\phi_p(x)$$
$$+\sum_{p=1}^{\infty}\left[\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^\gamma)G_p(s)\mathrm{d}s\right]\phi_p(x) \tag{3.8}$$

if and only if

$$\sum_{p=1}^{\infty}\left[\frac{g_p-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s}{E_{\gamma,1}(-\lambda_p T^\gamma)}\right]^2<\infty, \tag{3.9}$$

where $g_p=\left\langle g(x),\phi_p(x)\right\rangle$ and $G_p(s)=\left\langle G(x,s),\phi_p(x)\right\rangle$.

(b) *Let $g \in H^2(\Omega)$ and $G \in L^\infty(0,T;H^2(\Omega))$. Then problem has unique solution $u \in C([0,T];L^2(\Omega))\cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$.*

***Proof*:**

(a) Assume that the problem (1.2) has a unique solution u and the solution is given by (3.8). Then if we take $t=T$ in (3.8), we have

$$\begin{aligned}u(x,T)=g(x)\\ &=\sum_{p=1}^{\infty}\Bigg(E_{\gamma,1}(-\lambda_p T^\gamma)<u(x,0),\phi_p(x)>\\ &\quad+\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\Bigg)\phi_p(x).\end{aligned}$$

This implies that

$$\left\langle u(x,0),\phi_p(x)\right\rangle=\frac{\left\langle g(x),\phi_p(x)\right\rangle-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s}{E_{\gamma,1}(-\lambda_p T^\gamma)}. \tag{3.10}$$

Therefore

$$\begin{aligned}\sum_{p=1}^{\infty}\left[\frac{g_p-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s}{E_{\gamma,1}(-\lambda_p T^\gamma)}\right]^2&=\sum_{p=1}^{\infty}\left|\left\langle u(x,0),\phi_p(x)\right\rangle\right|^2\\ &\le\|u(.,0)\|^2_{L^2(\Omega)}<\infty.\end{aligned}$$

If (3.9) holds, then we define the function v as follows

$$v(x) = \sum_{p=1}^{\infty} \left[\frac{g_p - \int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^{\gamma}) G_p(s) ds}{E_{\gamma,1}(-\lambda_p T^{\gamma})} \right] \phi_p(x). \tag{3.11}$$

It is easy to see that $v \in L^2(\Omega)$. Now, we consider the following problem:

$$\begin{cases} D_t^{\gamma} u = \mathcal{L}u(x,t) + G(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = v(x), & x \in \Omega. \end{cases} \tag{3.12}$$

Since $v \in L^2(\Omega)$ and $G \in L^{\infty}(0,T;L^2(\Omega))$, we conclude that problem (3.12) has a unique solution which is given by

$$\begin{aligned} u(x,t) = \sum_{p=1}^{\infty} \Bigg(& E_{\gamma,1}(-\lambda_p t^{\gamma}) \langle v(x), \phi_p(x) \rangle \\ & + \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^{\gamma}) G_p(s) ds \Bigg) \phi_p(x). \end{aligned} \tag{3.13}$$

By following Theorem 2.1 in [2], we can easily prove that $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$. By (3.11) and (3.13), it is easy to see that

$$u(x,T) = \sum_{p=1}^{\infty} g_p \phi_p(x) = g(x). \tag{3.14}$$

This implies that u is the unique solution to the backward problem (1.2).

(b) Since $g \in H^2(\Omega)$ and $G \in L^{\infty}(0,T;H^2(\Omega))$, we have the following estimate

$$\begin{aligned} & \sum_{p=1}^{\infty} \left[\frac{g_p - \int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^{\gamma}) G_p(s) ds}{E_{\gamma,1}(-\lambda_p T^{\gamma})} \right]^2 \\ & \leq 2 \sum_{p=1}^{\infty} \frac{|< g, \phi_p(x) >|^2}{|E_{\gamma,1}(-\lambda_p T^{\gamma})|^2} + 2 \sum_{p=1}^{\infty} \frac{\left| \left\langle \int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^{\gamma}) G_p(s) ds, \phi_p(x) \right\rangle \right|^2}{E_{\gamma,1}^2(-\lambda_p T^{\gamma})}. \end{aligned} \tag{3.15}$$

The first term on the right-hand side of (3.15) is estimated as follows

$$2\sum_{p=1}^{\infty}\frac{\left|<g,\phi_p(x)>\right|^2}{\left|E_{\gamma,1}(-\lambda_pT^{\gamma})\right|^2}\le 2\sum_{p=1}^{\infty}\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\lambda_p^2\left|<g,\phi_p(x)>\right|^2$$
$$=2\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\|g\|^2_{H^2(\Omega)}, \qquad (3.16)$$

where we use Lemma 2.1. Now we estimate the second term on the right-hand side of (3.15).

$$2\sum_{p=1}^{\infty}\frac{\left|\left\langle\int_0^T(T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s,\phi_p(x)\right\rangle\right|^2}{E^2_{\gamma,1}(-\lambda_pT^{\gamma})}$$
$$\le 2\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\sum_{p=1}^{\infty}\lambda_p^2\left|\int_0^T(T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s\right|^2$$
$$\le 2M_2\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\|G\|^2_{L^{\infty}(0,T;H^2(\Omega))}. \qquad (3.17)$$

If we use (3.16) and (3.17) in (3.15), we prove (3.9). Applying Part (a) of this Theorem, we deduce that the problem (1.2) has unique solution u.

Now we prove that $u\in C([0,T];L^2(\Omega))\cap C((0,T];H^2(\Omega)\cap H^1_0(\Omega))$. By using (3.16) and (3.17) for all $t\in[0,T]$, we have

$$\|u(.,t)\|^2_{L^2(\Omega)}=\sum_{p=1}^{\infty}\left[\frac{E_{\gamma,1}(-\lambda_pt^{\gamma})}{E_{\gamma,1}(-\lambda_pT^{\gamma})}\left(g_p-\int_0^T(T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s\right)\right.$$
$$\left.+\int_0^t(t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^{\gamma})G_p(s)\mathrm{d}s\right]^2$$
$$\le 3\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\|g\|^2_{H^2(\Omega)}$$
$$+3M_2\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^-}\right]^2\|G\|^2_{L^{\infty}(0,T;H^2(\Omega))}++3M_1\|G\|^2_{L^{\infty}(0,T;L^2(\Omega))}. \qquad (3.18)$$

Since we know that

$$\sum_{p=1}^{\infty}\left[\frac{E_{\gamma,1}(-\lambda_pt^{\gamma})}{E_{\gamma,1}(-\lambda_pT^{\gamma})}\left(g_p-\int_0^T(T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s\right)\right]\phi_p(x)$$
$$+\sum_{p=1}^{\infty}\left[\int_0^t(t-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(t-s)^{\gamma})G_p(s)\mathrm{d}s\right]\phi_p(x) \qquad (3.19)$$

is convergent in $L^2(\Omega)$ uniformly for all $t \in [0, T]$, we conclude that $u \in C([0, T]; L^2(\Omega))$. Also by using the inequality $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$ and Lemma 2.1, we have the following estimate for all $t \in (0, T]$

$$\begin{aligned}
\|u(.,t)\|^2_{H^2(\Omega)} &= \sum_{p=1}^{\infty} \lambda_p^2 \Big[\frac{E_{\gamma,1}(-\lambda_p t^\gamma)}{E_{\gamma,1}(-\lambda_p T^\gamma)} \Big(g_p - \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^\gamma) G_p(s) \mathrm{d}s \Big) \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^\gamma) G_p(s) \mathrm{d}s \Big]^2 \\
&\le 3 \Big(\frac{B^+ T^\gamma}{B^- t^\gamma} \Big)^2 \sum_{p=1}^{\infty} \Big(\lambda_p^2 g_p^2 + \lambda_p^2 \Big| \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^\gamma) G_p(s) \mathrm{d}s \Big|^2 \Big) \\
&\quad + 3 \sum_{p=1}^{\infty} \lambda_p^2 \Big| \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^\gamma) G_p(s) \mathrm{d}s \Big|^2 \\
&\le 3 \Big(\frac{B^+ T^\gamma}{B^- t^\gamma} \Big)^2 \Big[\|g\|^2_{H^2(\Omega)} + M_1 \|G\|^2_{L^\infty(0,T;H^2(\Omega))} \Big] + 3M_1 \|G\|^2_{L^\infty(0,T;H^2(\Omega))}, \qquad (3.20)
\end{aligned}$$

which implies

$$\|u(.,t)\|_{H^2(\Omega)} \le \sqrt{3} \frac{B^+ T^\gamma}{B^- t^\gamma} \|g\|_{H^2(\Omega)} + \sqrt{3M_1} \Big(\frac{B^+ T^\gamma}{B^- t^\gamma} + 1 \Big) \|G\|_{L^\infty(0,T;H^2(\Omega))}. \qquad (3.21)$$

Since we know that

$$\begin{aligned}
&\sum_{p=1}^{\infty} \lambda_p \left[\frac{E_{\gamma,1}(-\lambda_p t^\gamma)}{E_{\gamma,1}(-\lambda_p T^\gamma)} \left(g_p - \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (T-s)^\gamma) G_p(s) \mathrm{d}s \right) \right] \phi_p(x) \\
&\quad + \sum_{p=1}^{\infty} \lambda_p \left[\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_p (t-s)^\gamma) G_p(s) \mathrm{d}s \right] \phi_p(x) \qquad (3.22)
\end{aligned}$$

is convergent in $L^2(\Omega)$ uniformly for all $t \in [\delta, T]$ with any given $\delta > 0$, we can see that $\mathcal{L}u \in C((0, T]; L^2(\Omega))$. This and (3.21) leads us to $u \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$. □

Remark 3.1: From the same techniques of the proof part (b) of Theorem 3.1, we can easily prove the following estimate:

$$\|u(.,t)\|_{L^2(\Omega)} \le \sqrt{3} \frac{B^+ T^\gamma}{B^- t^\gamma} \|g\|_{L^2(\Omega)} + \sqrt{3M_1} \Big(\frac{B^+ T^\gamma}{B^- t^\gamma} + 1 \Big) \|G\|_{L^\infty(0,T;L^2(\Omega))}. \qquad (3.23)$$

Hence, the backward problem is well posed for $t \in (0, T)$. This is quite different from $\gamma = 1$. Numerical experiments also confirm this. Our explanation to this phenomenon is that the fractional derivatives are hereditary functionals possessing a total memory of past states, so we could easily detect the previous status from its present information. But the state at $t = 0$ is an exception.

Here after we only focus on the case $t = 0$.

Theorem 3.2: *Let $g \in L^2(\Omega)$ and $G \in L^\infty(0,T;L^2(\Omega))$. Suppose that $u(.,0) \in H^m(\Omega)$ for any $m \geq 0$ and there exists a positive number M such that*

$$\|u(.,0)\|_{H^m(\Omega)} \leq M. \tag{3.24}$$

Then there holds

$$\|u(.,0)\|_{L^2(\Omega)} \leq PM^{\frac{1}{m+1}}, \tag{3.25}$$

where

$$P = P(m,\gamma,B^-,g,G) = 2^{\frac{m}{2m+2}} \left[\frac{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}{B^-}\right]^{\frac{m}{m+1}} \left[\|g\|^2_{L^2(\Omega)} + M_1\|G\|^2_{L^\infty(0,T;L^2(\Omega))}\right]^{\frac{m}{2m+2}}.$$

Proof: By using (3.8) and Hölder inequality $\sum_{k=1}^\infty |x_k y_k| \leq \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}\left(\sum_{k=1}^\infty |x_k|^q\right)^{1/q}$ for $p = m+1$ and $q = \dfrac{m+1}{m}$, we get

$$\begin{aligned}
&\|u(.,0)\|^2_{L^2(\Omega)} \\
&= \sum_{p=1}^\infty \frac{\left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^2}{|E_{\gamma,1}(-\lambda_p T^\gamma)|^2} \\
&= \sum_{p=1}^\infty \frac{\left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^{\frac{2}{m+1}}\left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^{\frac{2m}{1+m}}}{|E_{\gamma,1}(-\lambda_p T^\gamma)|^2} \\
&\leq J_1^{\frac{1}{m+1}} J_2^{\frac{m}{m+1}},
\end{aligned} \tag{3.26}$$

where $g_p = \langle g(x), \phi_p(x)\rangle$,

$$J_1 = \sum_{p=1}^\infty \frac{\left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^2}{|E_{\gamma,1}(-\lambda_p T^\gamma)|^{2m+2}},$$

$$J_2 = \sum_{p=1}^\infty \left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^2.$$

Using (3.10) and (3.17), J_1 is estimated as follows:

$$J_1 = \sum_{p=1}^\infty \frac{1}{|E_{\gamma,1}(-\lambda_p T^\gamma)|^{2m}} \frac{\left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^2}{|E_{\gamma,1}(-\lambda_p T^\gamma)|^2}$$

$$\leq \sum_{p=1}^{\infty}\left[\frac{\Gamma(1-\gamma)\left(T^{\gamma}+\frac{1}{\lambda_1}\right)}{B^{-}}\right]^{2m}\lambda_p^{2m}\left|\langle u(x,0),\phi_p(x)\rangle\right|^2$$

$$=\left[\frac{\Gamma(1-\gamma)\left(T^{\gamma}+\frac{1}{\lambda_1}\right)}{B^{-}}\right]^{2m}\|u(.,0)\|^2_{H^m(\Omega)}. \tag{3.27}$$

Using Lemma 3.1, J_2 is estimated as follows:

$$J_2=\sum_{p=1}^{\infty}\left[g_p-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s\right]^2$$

$$\leq 2\left[\sum_{p=1}^{\infty}g_p^2+\sum_{p=1}^{\infty}\left|\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^{\gamma})G_p(s)\mathrm{d}s\right|^2\right]$$

$$\leq 2\left[\|g\|^2_{L^2(\Omega)}+M_1\|G\|^2_{L^{\infty}(0,T;L^2(\Omega))}\right]. \tag{3.28}$$

By (3.26), (3.27), and (3.28), we obtain

$$\|u(.,0)\|^2_{L^2(\Omega)}\leq 2^{\frac{m}{m+1}}\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^{-}}\right]^{\frac{2m}{m+1}}\|u(.,0)\|^{\frac{2}{m+1}}_{H^m(\Omega)}\left[\|g\|^2_{L^2(\Omega)}+M_1\|G\|^2_{L^{\infty}(0,T;L^2(\Omega))}\right]^{\frac{m}{m+1}},$$

which implies

$$\|u(.,0)\|_{L^2(\Omega)}\leq 2^{\frac{m}{2m+2}}\left[\frac{\Gamma(1-\gamma)(T^{\gamma}+\frac{1}{\lambda_1})}{B^{-}}\right]^{\frac{m}{m+1}}\left[\|g\|^2_{L^2(\Omega)}+M_1\|G\|^2_{L^{\infty}(0,T;L^2(\Omega))}\right]^{\frac{m}{2m+2}}M^{\frac{1}{m+1}}. \tag{3.29}$$

Hence

$$\|u(.,0)\|_{L^2(\Omega)}\leq PM^{\frac{1}{m+1}}.$$

□

4. Tikhonov regularization under an a-priori parameter choice rule

Defining a linear operator $K:L^2(\Omega)\to L^2(\Omega)$ as follows:

$$Kf(x)=\sum_{p=1}^{\infty}E_{\gamma,1}(-\lambda_pT^{\gamma})\langle f,\phi_p\rangle\phi_p(x)=\int_{\Omega}k(x,\xi)f(\xi)\,\mathrm{d}\xi, \tag{4.1}$$

where $k(x,\xi)=\sum_{p=1}^{\infty}E_{\gamma,1}(-\lambda_pT^{\gamma})\phi_p(x)\phi_p(\xi)$. Since $k(x,\xi)=k(\xi,x)$, we know that the operator K is self-adjoint. Now we prove that the operator K is a compact operator. Let us consider the finite rank operators K_m defined by

$$K_mf(x)=\sum_{p=1}^{m}E_{\gamma,1}(-\lambda_pT^{\gamma})\langle f,\phi_p\rangle\phi_p(x). \tag{4.2}$$

Then, by Lemma 2.1, (4.1) and (4.2) we have

$$\begin{aligned}\left\|K_m f - Kf\right\|^2_{L^2(\Omega)} &= \sum_{p=m+1}^{\infty} \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2 \langle f, \phi_p\rangle^2 \\ &\le \left[\frac{B^+}{\Gamma(1-\gamma)} \frac{1}{1+\lambda_p T^\gamma}\right]^2 \sum_{p=m+1}^{\infty} \langle f, \phi_p\rangle^2 \\ &\le \left[\frac{B^+}{\Gamma(1-\gamma)} \frac{1}{1+\lambda_m T^\gamma}\right]^2 \|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.3}$$

Therefore $\|K_m - K\| \to 0$, as $m \to \infty$ in the sense of operator norm in $\mathbb{L}\left(L^2(\Omega); L^2(\Omega)\right)$. So K is a compact operator. We know that the singular values for the linear self-adjoint compact operator are

$$\sigma_p = E_{\gamma,1}(-\lambda_p T^\gamma), \tag{4.4}$$

and the corresponding eigenvectors are ϕ_p. From (4.1), the backward problem can be formulated as the following operator equation:

$$Ku(x,0) = h(x), \tag{4.5}$$

where

$$h(x) = \sum_{p=1}^{\infty} \left[g_p - \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}\Big(-\lambda_p (T-s)^\gamma\Big) G_p(s)\mathrm{d}s \right] \phi_p(x). \tag{4.6}$$

Since the problem (4.5) is ill-posed, we solve it by using Tikhonov regularization method which minimizes the following quantity in $L^2(\Omega)$:

$$\|Ku(.,0) - h\|^2_{L^2(\Omega)} + \alpha \|u(.,0)\|^2_{L^2(\Omega)},$$

where $\alpha > 0$ is a regularization parameter. The minimizer $u_\alpha(x,0)$ satisfies [19]

$$K^* K u_\alpha(.,0) + \alpha u_\alpha(.,0) = K^* h(x). \tag{4.7}$$

By singular value decomposition for compact self-adjoint operator K as in [20] , we have:

$$u_\alpha(.,0) = \sum_{p=1}^{\infty} \frac{E_{\gamma,1}(-\lambda_p T^\gamma)}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2} \left[g_p - \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}\Big(-\lambda_p (T-s)^\gamma\Big) G_p(s)\mathrm{d}s \right] \phi_p(x). \tag{4.8}$$

If the given data (g, G) is noised by (g^ϵ, G^ϵ), we have

$$u^\epsilon_\alpha(.,0) = \sum_{p=1}^{\infty} \frac{E_{\gamma,1}(-\lambda_p T^\gamma)}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2} \left[g^\epsilon_p - \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}\Big(-\lambda_p (T-s)^\gamma\Big) G^\epsilon_p(s)\mathrm{d}s \right] \phi_p(x). \tag{4.9}$$

In this work, we will deduce an error estimate for $\left\|u(.,0) - u^\epsilon_\alpha(.,0)\right\|_{L^2(\Omega)}$ and show convergence rate under a suitable choice of regularization parameters. It is clear that the total error can be decomposed into the bias and noise propagation as follows:

$$\left\|u(.,0)-u_\alpha^\epsilon(.,0)\right\|_{L^2(\Omega)} \le \|u(.,0)-u_\alpha(.,0)\|_{L^2(\Omega)} + \left\|u_\alpha(.,0)-u_\alpha^\epsilon(.,0)\right\|_{L^2(\Omega)}. \qquad (4.10)$$

We first give an error bound for the noise term.

Lemma 4.1: *If the noise assumption (1.3) holds, then*

$$\left\|u_\alpha(.,0)-u_\alpha^\epsilon(.,0)\right\|^2_{L^2(\Omega)} \le \frac{\|g^\epsilon-g\|^2_{L^2(\Omega)} + M_1\|G^\epsilon-G\|^2_{L^\infty(0,T;L^2(\Omega))}}{2\alpha}. \qquad (4.11)$$

***Proof*:** From (4.8), (4.9) and using the inequality $4ab \le (a+b)^2 \le 2(a^2+b^2)$, for any $a,b\in\mathbb{R}$ we get

$$\begin{aligned}
&\|u_\alpha^\epsilon(.,0)-u_\alpha(.,0)\|^2_{L^2(\Omega)} \\
&= \sum_{p=1}^{\infty}\left|\frac{E_{\gamma,1}(-\lambda_p T^\gamma)}{\alpha+\left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}\right]^2 \\
&\quad\times\left[(g_p^\epsilon-g_p)-\int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}\Big(-\lambda_p(T-s)^\gamma\Big)(G_p^\epsilon(s)-G_p(s))\mathrm{d}s\right]^2 \\
&\le \sum_{p=1}^{\infty}\frac{\left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}{4\alpha\left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}2 \\
&\quad\times\left[(g_p^\epsilon-g_p)^2+\left|\int_0^T (T-s)^\gamma E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)\Big[G_p^\epsilon(s)-G_p(s)\Big]\mathrm{d}s\right|^2\right] \\
&\le \frac{\|g^\epsilon-g\|^2_{L^2(\Omega)}+M_1\|G^\epsilon-G\|^2_{L^\infty(0,T;L^2(\Omega))}}{2\alpha} \le \frac{\epsilon^2(1+M_1)}{2\alpha}.
\end{aligned} \qquad (4.12)$$

Hence

$$\|u_\alpha^\epsilon(.,0)-u_\alpha(.,0)\|_{L^2(\Omega)} \le \frac{\epsilon\sqrt{1+M_1}}{\sqrt{2\alpha}}.$$

□

In order to obtain the error bound for the first term in (4.10), we usually need some a priori conditions. By Tikhonov's theorem, the operator K^{-1} is restricted to the continuous image of a compact set M. Thus, we assume $u(.,0)$ is in a compact subset of $L^2(\Omega)$. Here fter, we assume the condition (3.24).

Lemma 4.2: *If the a priori condition (3.24) holds, then*

$$\|u_\alpha(.,0)-u(.,0)\|_{L^2(\Omega)} \le \begin{cases} A_1 M\alpha^{\frac{m}{2}}, & \text{if } 0<m<2, \\ A_2 M\alpha, & \text{if } m\ge 2, \end{cases} \qquad (4.13)$$

where

$$A_1 = A_1(\gamma, m, B^-, T, \lambda_1) = \frac{1}{2}\left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^{-m}\left(\frac{2-m}{m}\right)^{\frac{2-m}{2}},$$

$$A_2 = A_2(\gamma, m, B^-, T, \lambda_1) = \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^{-2}\lambda_1^{2-m}.$$

Proof: By (4.8), we can deduce that

$$\begin{aligned}
&\|u_\alpha(.,0) - u(.,0)\|_{L^2(\Omega)}^2 \\
&= \sum_{p=1}^\infty \left(\frac{E_{\gamma,1}(-\lambda_p T^\gamma)}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2} - \frac{1}{E_{\gamma,1}(-\lambda_p T^\gamma)}\right)^2 \\
&\times \left[g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s\right]^2 \\
&= \sum_{p=1}^\infty \left[\frac{\alpha}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}\right]^2 \left[\frac{g_p - \int_0^T (T-s)^{\gamma-1}E_{\gamma,\gamma}(-\lambda_p(T-s)^\gamma)G_p(s)\mathrm{d}s}{E_{\gamma,1}(-\lambda_p T^\gamma)}\right]^2. \quad (4.14)
\end{aligned}$$

By (3.10) and (4.14), we have

$$\begin{aligned}
\|u_\alpha(.,0) - u(.,0)\|_{L^2(\Omega)} &= \sqrt{\sum_{p=1}^\infty \left[\frac{\alpha\lambda_p^{-m}}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}\right]^2 \lambda_p^{2m}\left\langle u(x,0), \phi_p(x)\right\rangle^2} \\
&\le \sup_{p\in\mathbb{N}} A(p)\sqrt{\sum_{p=1}^\infty \lambda_p^{2m}\left\langle u(x,0), \phi_p(x)\right\rangle^2} \\
&= \sup_{p\in\mathbb{N}} A(p)\|u(.,0)\|_{H^m(\Omega)}, \quad (4.15)
\end{aligned}$$

where

$$A(p) = \frac{\alpha\lambda_p^{-m}}{\alpha + \left|E_{\gamma,1}(-\lambda_p T^\gamma)\right|^2}.$$

From Lemma 2.1, $A(p)$ is bounded by

$$A(p) \le \frac{\alpha\lambda_p^{-m}}{\alpha + \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})\lambda_p}\right|^2} = \frac{\alpha\lambda_p^{2-m}}{\alpha\lambda_p^2 + \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^2}. \quad (4.16)$$

By letting $z = \lambda_p \geq z_0 = \lambda_1$ and $A = \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^2$ in Lemma 2.4, we obtain

$$\frac{\alpha\lambda_p^{2-m}}{\alpha\lambda_p^2 + \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma+\frac{1}{\lambda_1})}\right|^2} \leq \begin{cases} \frac{1}{2}\left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^{-m}\left(\frac{2-m}{m}\right)^{\frac{2-m}{2}}\alpha^{\frac{m}{2}}, & \text{if } 0 < m < 2, \\ \left|\frac{B^-}{\Gamma(1-\gamma)(T^\gamma + \frac{1}{\lambda_1})}\right|^{-2}\lambda_1^{2-m}\alpha, & \text{if } \quad m \geq 2. \end{cases} \tag{4.17}$$

Combining (3.24), (4.15), (4.16), (4.17), we conclude that

$$\|u_\alpha(.,0) - u(.,0)\|_{L^2(\Omega)} \leq \begin{cases} A_1 M\alpha^{\frac{m}{2}}, & \text{if } \quad 0 < m < 2, \\ A_2 M\alpha, & \text{if } \quad m \geq 2. \end{cases}$$

□

Theorem 4.1: *Assume that (1.3) and (3.24) hold, then the following convergence estimates are satisfied:*

(a) *If* $0 < m < 2$ *and choose* $\alpha = \left(\frac{\epsilon}{M}\right)^{\frac{2}{m+1}}$, *then*

$$\|u_\alpha^\epsilon(.,0) - u(.,0)\|_{L^2(\Omega)} \leq \left(A_1 + \sqrt{\frac{M_1+1}{2}}\right)M^{\frac{1}{m+1}}\epsilon^{\frac{m}{m+1}}.$$

(b) *If* $m > 2$ *and choose* $\alpha = \left(\frac{\epsilon}{M}\right)^{\frac{2}{3}}$, *then*

$$\|u_\alpha^\epsilon(.,0) - u(.,0)\|_{L^2(\Omega)} \leq \left(A_2 + \sqrt{\frac{M_1+1}{2}}\right)M^{\frac{1}{3}}\epsilon^{\frac{2}{3}}.$$

Proof: The proofs are easily obtained by (4.10), Lemmas 4.1, and 4.2. □

5. Numerical experiments

In this section, we implement the presented regularization method on two examples. In these examples, we take $d = 2$ and $T = 1$. The nosiy data (g^ϵ, G^ϵ) is obtained as follows:

$$\begin{aligned} G^\epsilon(.,.,.) &= G(.,.,.) + \epsilon(2rand() - 1), \\ g^\epsilon(.,.) &= g(.,.) + \epsilon(2rand() - 1). \end{aligned} \tag{5.1}$$

We can easily verify that (1.3) is satisfied. The regularization parameter α depends on ϵ. Hence, we use the Matlab command fzero to find $\alpha(\epsilon)$. Also we take the regularization parameter α as $\alpha_{pri} = \left(\frac{\epsilon}{M}\right)^{\frac{2}{3}}$, where M is the a priori condition computed by $\|u(.,.,0)\|_{L^2(\Omega)} \leq M$. We compute the Mittag–Leffler function by using the algorithm given in [21]. The examples are considered in the following form:

$$\begin{cases} \frac{\partial^\gamma u(x,y,t)}{\partial t^\gamma} = u_{xx}(x,y,t) + u_{yy}(x,y,t) + G(x,y,t), & (x,y) \in \Omega, t \in (0,1], \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega, t \in (0,1], \\ u(x,y,T) = g(x,y), & (x,y) \in \Omega, \end{cases} \tag{5.2}$$

where $\Omega = (0,\pi) \times (0,\pi)$. By (3.8) and (4.9), the exact and its regularized solutions are found as follows:

$$u(x,y,0) = \frac{4}{\pi^2}\sum_{p=1}^{N}\sum_{q=1}^{N}\frac{1}{E_{\gamma,1}(-\lambda_{pq}T^{\gamma})}\left(g_{pq} - \int_0^T \Theta(s,\gamma,\lambda_{pq})G_{pq}(s)\mathrm{d}s\right)\phi_{pq}(x,y),$$
$$u^{\epsilon}_{\alpha(\epsilon)}(x,y,0) = \frac{4}{\pi^2}\sum_{p=1}^{N}\sum_{q=1}^{N}\frac{1}{[E_{\gamma,1}(-\lambda_{pq}T^{\gamma})]^2+\alpha(\epsilon)}\left(g^{\epsilon}_{pq} - \int_0^T \Theta(s,\gamma,\lambda_{pq})G^{\epsilon}_{pq}(s)\mathrm{d}s\right)\phi_{pq}(x,y), \tag{5.3}$$

where

$$\begin{aligned}\lambda_{pq} &= p^2+q^2,\\ \phi_{pq}(x,y) &= \sin(px)\sin(qy),\\ \Theta(s,\gamma,\lambda_{pq}) &= (T-s)^{\gamma-1}E_{\gamma,\gamma}(-(p^2+q^2)(T-s)^{\gamma})\end{aligned} \tag{5.4}$$

and N is enough large number (truncation number).

In our computations, we take $P = Q = L = 101$ to generate temporal and spatial discretizations as follows:

$$\begin{aligned} x_i &= i\triangle x, \triangle x = \frac{\pi}{P}, \quad i = 0\cdots P,\\ y_j &= j\triangle y, \triangle y = \frac{\pi}{Q}, \quad j = 0\cdots Q,\\ t_k &= k\triangle t, \triangle t = \frac{1}{L}, \quad k = 0\cdots L.\end{aligned}$$

We also set $u^{\epsilon}_{\alpha(\epsilon)}(x_i,y_j,0) = u^{\alpha(\epsilon),\epsilon}_{i,j,0}$ and $u(x_i,y_j,0) = u_{i,j,0}$ to construct two vectors contain all discrete values of $u^{\epsilon}_{\alpha(\epsilon)}$ and f. We denote them by $\Lambda^{\epsilon}_{\alpha(\epsilon)}$ and Ξ, respectively. These two matrices are used in showing our results.

$$\Lambda^{\epsilon}_{\alpha(\epsilon)} = \begin{bmatrix} u^{\alpha(\epsilon),\epsilon}_{(0,0,0)} & u^{\alpha(\epsilon),\epsilon}_{(0,1,0)} & \cdots & u^{\alpha(\epsilon),\epsilon}_{(0,Q-1,0)} & u^{\alpha(\epsilon),\epsilon}_{(0,Q,0)}\\ u^{\alpha(\epsilon),\epsilon}_{(1,0,0)} & u^{\alpha(\epsilon),\epsilon}_{(1,1,0)} & \cdots & u^{\alpha(\epsilon),\epsilon}_{(1,Q-1,0)} & u^{\alpha(\epsilon),\epsilon}_{(1,Q,0)}\\ u^{\alpha(\epsilon),\epsilon}_{(2,0,0)} & u^{\alpha(\epsilon),\epsilon}_{(2,1,0)} & \cdots & u^{\alpha(\epsilon),\epsilon}_{(2,Q-1,0)} & u^{\alpha(\epsilon),\epsilon}_{(2,Q,0)}\\ \cdots & \cdots & \ddots & \cdots & \cdots\\ u^{\alpha(\epsilon),\epsilon}_{(P,0,0)} & u^{\alpha(\epsilon),\epsilon}_{(P,1,0)} & \cdots & u^{\alpha(\epsilon),\epsilon}_{(P,Q-1,0)} & u^{\alpha(\epsilon),\epsilon}_{(P,Q,0)} \end{bmatrix} \in \mathbb{R}^{P+1}\times\mathbb{R}^{Q+1},$$
$$\Xi = \begin{bmatrix} u_{(0,0,0)} & u_{(0,1,0)} & \cdots & u_{(0,Q-1,0)} & u_{(0,Q,0)}\\ u_{(1,0,0)} & u_{(1,1,0)} & \cdots & u_{(1,Q-1,0)} & u_{(1,Q,0)}\\ u_{(2,0,0)} & u_{(2,1,0)} & \cdots & u_{(2,Q-1,0)} & u_{(2,Q,0)}\\ \cdots & \cdots & \ddots & \cdots & \cdots\\ u_{(P,0,0)} & u_{(P,1,0)} & \cdots & u_{(P,Q-1,0)} & u_{(P,Q,0)} \end{bmatrix} \in \mathbb{R}^{P+1}\times\mathbb{R}^{Q+1}.$$

The relative error between the exact and its regularized solutions is computed by the following formula:

$$e = \frac{\sqrt{\sum_{i=0}^{P}\sum_{j=0}^{Q}\left|u^{\epsilon}_{\alpha(\epsilon)}(x_i,y_j,0)-u(x_i,y_j,0)\right|^2}}{\sqrt{\sum_{i=0}^{P}\sum_{j=0}^{Q}\left|u(x_i,y_j,0)\right|^2}}.$$

Example 1: In the first example, we take $G(x,y,t)$ and $g(x,y)$ as follows:

$$\begin{aligned} G(x,y,t) &= \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}e^t \sin(xy)xy(\pi-x)(\pi-y) \\ &\quad + e^t(x^2+y^2)\sin(xy)xy(\pi-x)(\pi-y) - 2e^t(x^2+y^2)(\pi-x)(\pi-y) \\ &\quad + \Big[x(\pi-x)+y(\pi-y)\Big]\Big(2e^t\sin(xy)+2xye^t\cos(xy)\Big), \\ g(x,y) &= e\sin(xy)xy(\pi-x)(\pi-y). \end{aligned}$$

We note that the exact solution of the problem (5.2) for the given functions above is $u(x,y,0) = xy(\pi-x)(\pi-y)\sin(xy)$. In practice, it is very difficult to obtain the value M for the a priori parameter choice rule without having an exact solution. We take $M = 304482$ in $\|u(.,.,0)\|_{H^2(\Omega)} \le M$. The relative and absolute errors are shown in Table 1 from which we can see that the error is decreasing as the level of noise becomes smaller. The convergence order is found to be 0.66. This is consistent with our convergence estimate. For the convergence rate, we use the following definition:

$$\text{Convergence rate} = \log_{10}\frac{e(f,10\epsilon)}{e(f,\epsilon)}.$$

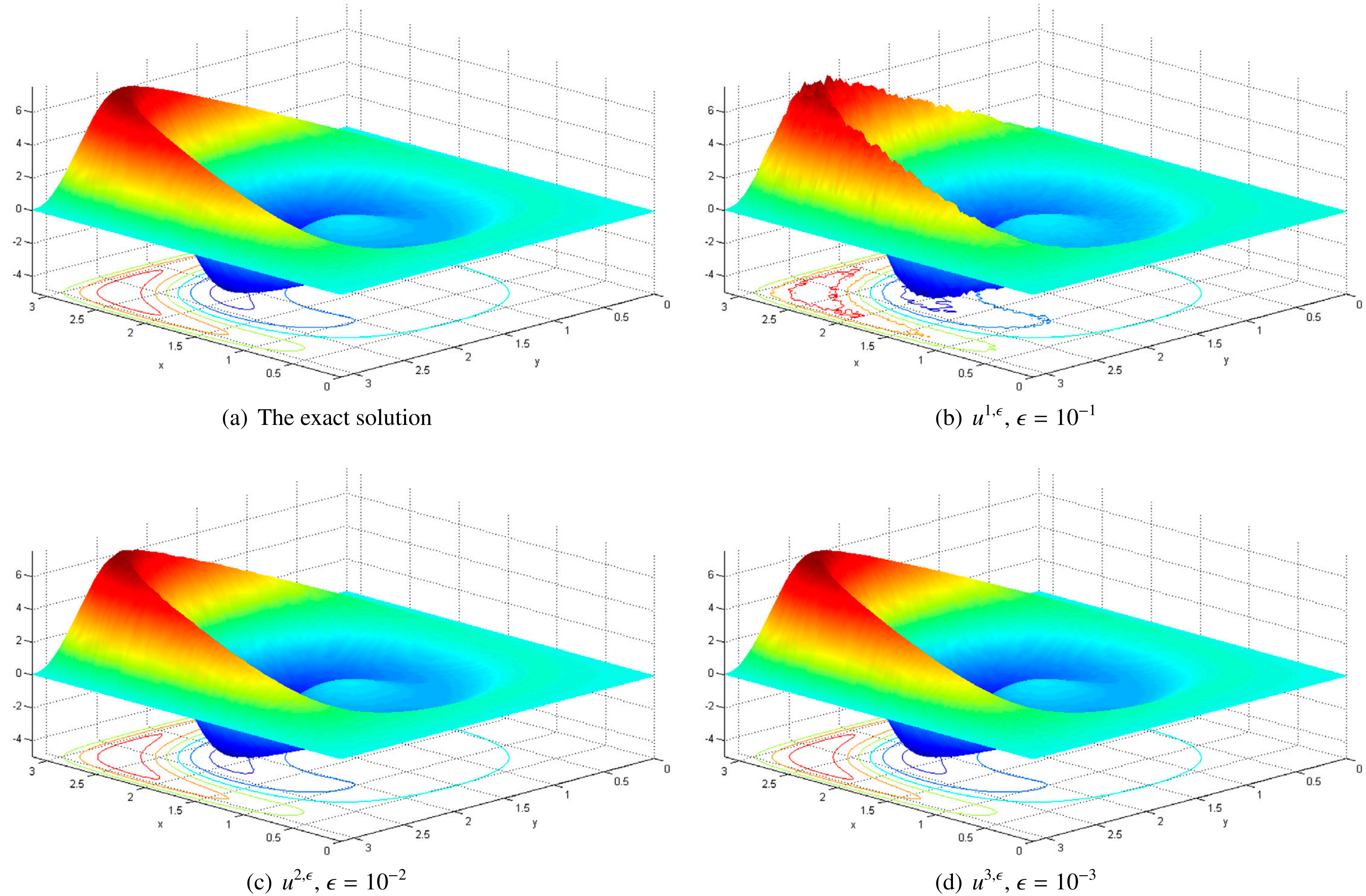
The numerical results under the a priori parameter choice rule when $\epsilon = 10^{-1}, \epsilon = 10^{-2}$, and $\epsilon = 10^{-3}$ are showed in Figure 1 with $\alpha_{pri_1} = 4.76E-05, \alpha_{pri_2} = 1.03E-05$, and $\alpha_{pri_3} = 2.21E-06$, respectively. We can see that the numerical results are in very good agreement with the exact solution.

Example 2: In the second example, we take $G(x,y,t)$ and $g(x,y)$ as follows:

$$\begin{aligned} G(x,y,t) &= \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}e^t\sin(xy)\sin((\pi-x)(\pi-y)) \\ &\quad + \Big[-e^t(x^2+y^2)\sin(xy) - e^t\cos(xy)[x(\pi-x)+y(\pi-y)]\Big]\sin((\pi-x)(\pi-y)) \\ &\quad - \Big[e^t[(\pi-x)^2+(\pi-y)^2]\sin(xy) - e^t\Big[x(\pi-x)+y(\pi-y)\Big]\cos(xy)\Big] \\ &\quad \times\cos((\pi-x)(\pi-y)). \\ g(x,y) &= e\sin(xy)\sin((\pi-x)(\pi-y)). \end{aligned} \tag{5.5}$$

We note that the exact solution of the problem (5.2) for the given functions above is $u(x,y,0) = \sin(xy)\sin((\pi-x)(\pi-y))$. We take $M = 315888$ in $\|u(.,.,0)\|_{H^2(\Omega)} \le M$. The relative and absolute errors are shown in Table 1. The numerical results under the a priori parameter choice rule when $\epsilon = 10^{-1}, \epsilon = 10^{-2}$, and $\epsilon = 10^{-3}$ are showed in Figure 2 with $\alpha_{pri_1} = 4.64E-05$, $\alpha_{pri_2} = 1.00E-05$, and $\alpha_{pri_3} = 2.16E-06$, respectively.

Table 1 shows the relative errors between the exact solution and its regularized solution for $\gamma = 0.2, \gamma = 0.5$, and $\gamma = 0.8$, where ex_1 denotes the error between the exact and its regularized solutions in Example 1 and ex_2 denotes the error between the exact and its regularized solutions in Example 2. As it is seen in Table 1, the regularized solution converges very well to the exact solution



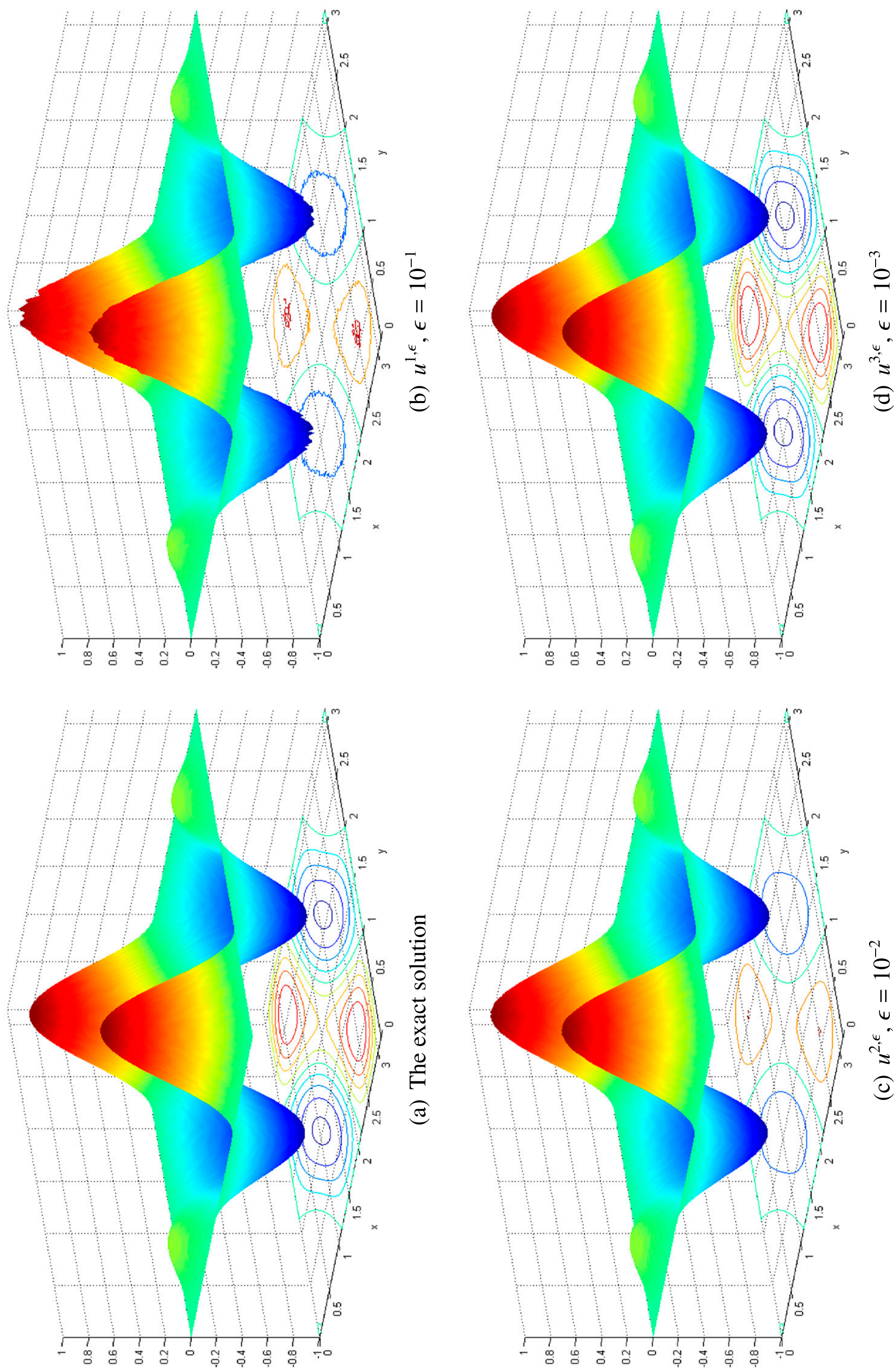
(a) The exact solution

(b) $u^{1,\epsilon}$, $\epsilon = 10^{-1}$

(c) $u^{2,\epsilon}$, $\epsilon = 10^{-2}$

(d) $u^{3,\epsilon}$, $\epsilon = 10^{-3}$

Figure 1. A comparison between the exact and its regularized solutions for the a priori parameter choice rule method in Example 1 for $\gamma = 0.2$.



(a) The exact solution

(b) $u^{1,\epsilon}$, $\epsilon = 10^{-1}$

(c) $u^{2,\epsilon}$, $\epsilon = 10^{-2}$

(d) $u^{3,\epsilon}$, $\epsilon = 10^{-3}$

Figure 2. A comparison between the exact and its regularized solutions for the a priori parameter choice rule in Example 2 for $\gamma = 0.2$.

Table 1. The relative root mean square error between the exact and regularized solutions of two examples for different values of ϵ.

	$\gamma = 0.2$		$\gamma = 0.5$		$\gamma = 0.8$	
ϵ	ex_1	ex_2	ex_1	ex_2	ex_1	ex_2
1E-01	9.86E-01	9.85E-01	9.94E-01	9.94E-01	9.99E-01	9.99E-01
1E-02	9.39E-01	9.37E-01	9.73E-01	9.72E-01	9.96E-01	9.96E-01
1E-03	7.64E-01	7.59E-01	8.84E-01	8.81E-01	9.81E-01	9.81E-01
1E-04	3.97E-01	3.91E-01	6.12E-01	6.06E-01	9.17E-01	9.15E-01
1E-05	1.19E-01	1.16E-01	2.42E-01	2.37E-01	6.97E-01	6.92E-01
1E-06	2.76E-02	2.69E-02	6.21E-02	6.06E-02	3.18E-01	3.12E-01
1E-07	6.00E-03	5.90E-03	1.39E-02	1.36E-02	8.75E-02	8.55E-02
1E-08	1.30E-03	1.27E-03	3.00E-03	2.95E-03	1.99E-02	1.95E-02
1E-09	2.82E-04	2.75E-04	6.51E-04	6.36E-04	4.30E-03	4.24E-03
1E-10	6.07E-05	5.92E-05	1.40E-04	1.37E-04	9.38E-04	9.16E-04

once ϵ tends to 0. It also confirms again the regularized solution converges relatively good to the exact solution in our proposed method.

6. Concluding remarks

We have studied a backward problem for an inhomogeneous time-fractional diffusion equation with variable coefficients in a general bounded domain. The backward problem, which means to recover the initial state for some slow diffusion process from its present status, is very hard to solve due to the nonlocal property of the fractional derivative and the irreversibility of the time. The backward problem is ill-posed and we propose a regularizing scheme by using Tikhonov method. By using a priori regularization parameter choice rule, we prove the convergence rate for the regularized solution. Numerical examples illustrate applicability and high accuracy of the proposed method. Although this paper focuses on an a-priori choice of the regularization parameter, there is usually a defect in any a-priori method i.e. the a-priori choice of the regularization parameter depends obviously on the a-priori bound M of the solution. In fact, the a-priori bound M cannot be known exactly in practice, and working with a wrong constant M may lead to a poor regularized solution. In the upcoming studies, we will mainly consider the a-posteriori choice of a regularization parameter for the mollification method. Using the discrepancy principle we will provide a new posteriori parameter choice rule. These are subjects of the planned studies by the authors of this paper.

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